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Use of Visual Representations Mediated by Talk to Support Students' learning in Mathematics

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Abstract

This study looks at how visual representations mediated by teachers' orchestration of classroom talk can be used to support students' learning of Mathematics. Lessons were planned with a focus on the use of 'talk moves' (Chapin, O'Connor & Anderson, 2013) that mediated the teachers' use of visual representations during lessons on the topic of Permutation and Combination (P&C). A study was done in the form of teacher-guided class discussions where teachers posed questions to guide students in mathematical reasoning tasks with the aim of getting students to talk about Mathematics in such a way that their understanding of concepts was revealed. Meaningful questions were used to elicit from students how a problem was solved and why a particular method was chosen. Useful strategies employed during the observed lesson were discussed during a teachers' focus group discussion. The students' and teachers' feedback was analysed and the pedagogical implications were discussed.

Introduction

According to the *Mathematics Syllabus* (Ministry of Education, 2015), the key mathematical skills required for effective learning of Mathematics include the ability to:

- analyse mathematical situations,
- construct logical arguments,
- use mathematical language to express mathematical ideas and arguments precisely, concisely and logically, and
- see and make linkages among mathematical ideas.

O'Halloran (2005) discussed how the use of linguistic, symbolic and visual representations can, in themselves, provide a means and resource for understanding Mathematics. In the context of problem solving – which is one of the core competencies required of students in the A level Mathematics syllabus, these representations translate into problem solving heuristics (e.g. drawing diagrams, developing systematic listings and classifications, using a simpler problem) that guide students in understanding the problem and devising a solution plan (Polya & Conway, 2004). However representations and problem solving heuristics only handle one aspect of students' learning. Other areas, such as clarifying misconceptions through questioning and deepening learning through critiquing one's own and others' ideas can better be addressed by exploring the conduct of classroom talk during lessons. Thus, by studying the use of visual representations mediated by classroom talk, we hope to find out how students' learning of Mathematics and problem solving can be better supported.

Literature Review

The English Language Institute of Singapore (ELIS) has adapted the work of Michaels and O'Connor (2015) and Zwiers and Crawford (2011) in developing a useful framework on teachers' 'talk moves', that is, 'strategic ways of asking questions and inviting participation in classroom conversations' (Chapin, O'Connor, & Anderson, 2013, p. 11) that can support the orchestration of classroom talk for productive academic discussion. There are five focus areas of talk moves (Fig. 1):

Focus Area		Talk Moves
1	Voicing and clarifying students' ideas	• Seek clarification • Revoice for verification
2	Listening closely to other students	• Ask student to restate another students' contribution
3	Deepening individual students' reasoning	• Probe for reasoning or evidence • Challenge students' statement or assumption
4	Engaging with each other's reasoning	• Elicit students' views on other students' ideas • Guide students to build on other students' contribution
5	Consolidating discussion points (in extended discussion)	• Get students to summarise/consolidate

Fig. 1: *Teacher Talk Moves from ELIS's Opening Up Talk for Learning in Subject Classrooms course*

In a Singapore secondary Mathematics study, Hogan, Chan, Rahim, Towndrow, and Kwek (2012) noted that the 'key pivots of the discursive regime in Singaporean Mathematics classroom' were 'procedural and explanatory talk' (p. 26). Our study further contextualizes this finding in our examination of how teachers draw out evidence of students' reasoning during class discussions. In a more recent report, Koay (2016) espoused the benefits of productive classroom talk, and gave a detailed illustration of how specific talk moves can be used by Singapore primary Mathematics teachers to deepen students' learning during tasked-based lessons. Indeed, talk moves can also be employed to promote mathematical thinking and student-centred learning during group work when there is a need to:

- anticipate student responses to cognitively demanding mathematical tasks;
- monitor students' responses when they are exploring tasks;
- select particular students to present, discuss and summarise their mathematical responses;
- sequence students' responses purposefully;
- help students connect their responses with key mathematical ideas (Stein, Engle, Smith, & Hughes, 2008).

However, current research has been limited to primary and secondary level Mathematics and little has been done to look at how the use of talk can be used to bring out productive Mathematics discussions in the junior college classroom. Hence our study seeks to explore how the conscious use of talk moves can help to develop students' mathematical thinking and problem solving abilities in handling the rigours of A-level Mathematics. For this purpose, we have selected the topic of Permutation and Combinations (P&C) – which involves the selection and arrangement of objects taken from a collection – as this has been found to be one area that students experienced great difficulty in applying the formulas correctly to the contextual problems that are presented. This is true even though it is possible to mitigate the difficulties that arise through the use of visual representations, as supported by a study conducted by Hegarty & Kozhevnikov (1999), where the use of visual representations was found to have resulted in positive problem solving outcomes – a conclusion which agrees with Polya and Conway's (2004) well-established judgment on the use of visuals as a useful problem solving heuristic. However, it is sad to note that the effectiveness of visuals in problem solving is somewhat marred by students' dislike for visual methods (Eisenberg & Dreyfus, 1991) despite them facing difficulties in tackling word problems (Gagne, 1983). Neverthless, the rich research supporting the use of spatial visualisation in Mathematics education has earned it an enduring place in the local Mathematics syllabus as it was featured as one of the necessary skills listed in the Singapore Mathematics Framework (Ministry of Education, 2015) that has been in use since 1988. In fact, teaching frameworks that are widely used in local mathematical thinking today, such as the the Model Method (Kho, 1987) and the Paul's Wheel of Reasoning (Paul, 1992) are just some successful strategies for using visual representations that can, indeed, go a long way in helping students to make sense of mathematical problems and solve them.

We have chosen to focus on the use of visual representations mediated by talk as talk moves can generate productive classroom discussions by enabling the teacher to guide the students into revealing the details of their reasoning, which are otherwise not clearly reflected in the solutions or captured in the visuals. Our project focus concurs with the findings reported in an earlier work by Presmeg (2006), where 'concrete imagery needs to be coupled with rigorous analytical thought processes to be effectively used in Mathematics' (p. 209).

Hence, in order to understand how teachers can support students in the learning of Mathematics, the research questions of this study are as follows:

i) How are teachers using visual representations mediated by talk to support learning in Mathematics?
ii) What are teachers' and students' perspectives on the strengths and challenges faced in the use of visual representations mediated by talk in Mathematics?
iii) What are the pedagogical implications of using visual representations mediated by talk in Mathematics?

Methodology

The research design for this study draws on the Design Research methodology (Collins, Joseph, & Bielaczyc, 2004), which investigates whether and why an intervention works in a certain context (Plomp & Nieveen, 2007). The intervention here refers to the use of specific visual representations, e.g. Paul's Wheel of Reasoning (Paul, 1992), graphic organizers and diagrams, mediated by specific talk moves (Chapin, O'Connor, & Anderson, 2013). During the intervention process, strategic questions in the form of talk moves can scaffold students' thinking, drawing connections and making logical links.

The research study comprised two lesson cycles. The study was not longitudinal, even though there were some overlaps in the students chosen for each of the cycles. Students who participated in this study were from the low ability group, selected based on their year end examination results. The first cycle involved 30 JC1 students from four classes and the second, conducted around six months later, consisted of 25 JC2 students from 17 classes. For each cycle, the participants were given a P&C problem worksheet that they had to attempt on the spot, directly before the class discussions, the focus of this study.

The study team members who attended took on dual roles as table group facilitators and lesson observers. The observation by team members was especially important as they were tasked with the role of capturing the teacher's actions, from the managing of the classroom dynamics, and explicit instruction to the monitoring of how the students responded to the various intervention strategies. With regard to research design, the single group pre- and post-test design was chosen because of its high adaptability to changes in teacher and student profiles. This provided the team greater flexibility to explore other areas of strengths and weaknesses, and to make the necessary adjustments and modifications where required. While the earlier cycle adhered strictly to the single group pre- and post-test research design, the second cycle was an adaptation in that the pre- and post-test components were replaced by the P&C questions found in the earlier March Common Test and the subsequent Mid-Year Assessment respectively. The number of teachers directly involved in instruction also differed from one cycle to the other. In the first cycle, the intervention took the form of a lesson scheduled outside the normal curriculum time. It was conducted by three teachers (Teacher A, Teacher B and Teacher C) whereas the intervention during the second cycle was conducted during the topical remedial programme for low ability students and therefore helmed by only one teacher i.e. Teacher B. Due to timetabling constraints, the teacher facilitators cum observers who were present during the first intervention were also different from those who attended the second intervention.

To facilitate the analysis, the lessons were recorded in the form of videos and transcripts so as to better capture the extent to which the different types of talk moves were used to promote students' learning. The lesson transcripts were then subsequently analysed using a qualitative interpretative approach to determine the frequency of types of talk moves used by each teacher.

Results from the pre- and post-tests (Appendix 1) indicated how much the students had learnt from the lesson. Other sources of students' data such as the worksheets and feedback also provided timely information for teachers on the struggles, difficulties and learning gaps faced by the students. Teachers' feedback and perspectives were also gathered from the teachers' focus group discussion and individual reflections. The teachers could use this information to improve subsequent phases of implementation and also gain more insight into their own teaching practices.

Planning Phase

The topic of P&C was chosen for this study as this is one area of A-Level Mathematics that many students encounter difficulty with. Furthermore, P&C questions are usually word problems involving the selection and/or arrangement of objects, which can thus offer rich opportunities for the use of visual representations. The intervention strategies adopted were *Talk Moves*, *Visual Representations* and *Teachers' Facilitated Discussion* for the session.

During the lesson preparation stage, the team discussed how to frame specific questions using specific talk moves (Fig. 1) in order to make student mathematical thinking more visible so that students' misconceptions could be accurately diagnosed. The nature of questions in the chosen topic of Permutation and Combination (P&C) is complex for many students because the vocabulary used in these questions is also used in everyday English. This causes confusion for students who lack proficiency in Mathematical literacy, and hence may lead to increasing the possibility of misinterpretations setting in. To help teachers in this aspect, the lesson plan included useful and well-constructed question prompts (Fig. 2) that could be used to complement the teaching of

<u>Lesson Questions Discussion</u>

Q1 *Concepts: Multiplication Principle, Principle of Inclusion and Exclusion.*

The digits 1, 2, 3, 8, 9 are used to form a 7-digit number that satisfy one of these guidelines, or both:

(i) The first two digits must be 1 and 2, in some order;

(ii) The last two digits must be 8 and 9, in some order.

How many different 7-digit numbers can be formed?

Explanation & Possible Errors	**Prompt & Guiding Questions** **(Teacher will ask the following questions only if necessary, otherwise, try to ask questions that guide students to explain their thought processes to the class. Focus on 'how do you think of…', 'what make you think of …' types of questions)**
<u>Case III Explanation</u> The first two digits must be 1 and 2 →2! Permutations for the first two positions. <u>AND</u> The last two digits must be 8 and 9 → 2! Permutations for the last two positions. <u>AND</u> No restrictions on the choice of digits for middle three positions → repetition is allowed →each position can be filled by any one of the 5 possible digits → $^5C_1$ for each position. *Apply Multiplication Principle (<u>AND</u>)* **<u>Possible Errors</u>** Students may not be able to visualize the exclusion of double counting events. To help them see, ask them to consider the 7-digit number <u>12</u>333<u>89</u> (satisfies both conditions)	1. How does a possible 7-digit number that satisfies guideline (i) looks like? Expect students to give examples 2. What do your examples tell you about the situation here as compared to those given under known results? 3. In particular, ask yourself these 3 questions: a. Any arrangements involved? What are the 'objects' and what are the 'boxes'? *Objects are digits, boxes are the spaces.* b. Is the selection/arrangement done with or without replacement? *With replacement.* c. Are the 'objects' here chosen from a set of all distinct objects? *Yes.* d. What technique works best here? *Box Method.* **[Does every class know what the Box Method is? Here you should focus on getting students to 'see' the boxes.]** 4. What is meant by the phrase 'satisfy one or both'? Can anyone give me an example of how you can use this phrase in another context? → Link to idea of unions→ Can you recall any formula associated with unions? $n(A\cup B)=n(A)+n(B)-n(A\cap B)$

Fig. 2: Lesson Plan – Possible Errors, Explanations, and Question Prompts

problem solving using relevant visual diagrams, which included box diagrams (Fig. 3a), Venn diagrams (Fig. 3b) and direct listing of possibilities (Fig. 3c). In the discussions, individual team members also shared how they would teach the lesson and explain students' mistakes that were particularly difficult to put across. These pre-empted errors were also captured in the lesson plan (Fig. 2).

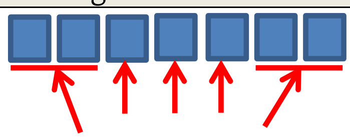
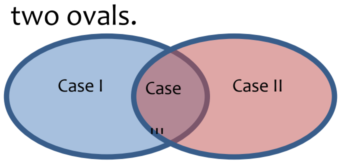
(a) Box Diagram	(b) Venn Diagram	(c) Direct Listing
 $2!\times\binom{5}{1}\times\binom{5}{1}\times\binom{5}{1}\times 2!$	Case I and Case II are the blue and red whole ovals respectively, while Case III is the intersection of the two ovals.  	Rearranging MATHEMATICS, we have MM AA TT H E I C S, or M A T H E I C S 2 2 2 1 1 1 1 1

Fig. 3: Visual Representations

Implementation Phase

The intervention was conducted in two three-hour sessions in November 2015 and May 2016 respectively. Three problems were selected for discussion during each of the research lessons. In accordance with the single group pre-test post-test research design, the same set of questions was used for both the pre- and post-tests so as to eliminate the possibility of inaccurate measurement due to the change in the questions. The pre- and post-tests were conducted immediately before the start and after the end of the research lesson respectively so as to filter off any other possible influences that might have impacted students' learning other than those that were derived from the lessons. In line with the aims of the project, all the teachers employed a mix of question prompts, and symbolic and visual cues to guide students in unpacking and understanding the mathematical situation posed. The lessons took the form of classroom discussions, where solution approaches were developed with the help of visual representations such as boxes, Venn diagrams and the systematic listing of cases (Fig. 3).

The research lesson started with a teacher-guided question conducted by Teacher A (Fig. 4), and the student participants were asked to do a think-pair-share.

Teacher-Guided Question

Can you recall how you solve this problem?

'Find the number of different code words that can be formed using the letters from the word EQUILIBRIUM if the consonants are always in alphabetical order from left to right.'

Write down your method and solution below and explain to your friend how you get your answer.

Fig. 4: The 'EQUILIBRUM' problem

After allocating some time for the students to work through the problem individually and in pairs, Teacher A asked a series of open ended questions (Fig. 5) to ease students into the routine of asking and answering their own questions.

Speaker	Lesson Transcript	Talk Moves
Teacher A	Which part of the question tells you to use all the letters? Yes, Nasri? You tell them.	
Student X	That one, [Inaudible].	
Teacher A	[Pointing to a word on the whiteboard] Here? Using letters.	
Student X	Yes.	
Teacher A	Ok, **if they want you to not use all the letters what would they say?**	**Probe for reasoning or evidence**
Student X	Letter, or bracket 's'.	
Teacher A	Use letter, singular?	
Student X	No, bracket 's', yes.	
Teacher A	Do you think so? Nobody will form a code word with one letter, ok. How many letters? They may add in some phrasing. **What would they add in? For example?**	**Probe for reasoning or evidence**

Speaker	Lesson Transcript	Talk Moves
Students	Some of the letters.	
Teacher A	Some of the letters, ok that could be one way it could be phrased, using some of the letters, from the word equilibrium, this could be one way, if they don't want to take all the letters. **Any other possibilities? Whereby you don't use all the letters? Maybe only use, five or six of the letters? How would the question be phrased? Anyone?**	**Challenge a student's statement or assumption**

Fig. 5: Excerpts of transcript with open-ended questions (bold)

During this stage of the lesson, the other research team members took on the role of group facilitators to address specific concerns raised by some participants within the groups. This was a crucial responsibility that played a vital role in levelling up the ability of each group to match that of the class. Questions raised by the students that were worthy of further discussion were also shared with the rest (Fig. 6).

Speaker	Lesson Transcript	Talk Moves
Teacher A	Ok, why is it important to count the number of repeated letters? Ok please remember when you are finding the number of ways to arrange the letters, this has to be taken into consideration. Okay, using letters from the word, these are some phrasing that you have to be very careful, so you have this [underlining the word on whiteboard] alphabetical order. Then there's this 'using letters from the word', so there's one question that some of you have asked. What did you ask just now? [Pointing] This group right? **What did you say?**	**Seek Clarification**
Student X	Must we use all the letters?	
Teacher A	**Must we use all the letters? From the word equilibrium? Why? Yes or no?**	**Probe for reasoning or evidence**
Students	Yes	

Fig. 6: Excerpts of transcript that show how the teacher directed Student X's question (bold) to the rest of the class

Teacher A used the whiteboard to record any key points that were surfaced in the discussion and to capture the development of these ideas as the discussion progressed (Fig. 7). She also inserted sufficient pauses and repeated the same question occasionally during the discussion to allow students the opportunity to clarify their questions with one another.

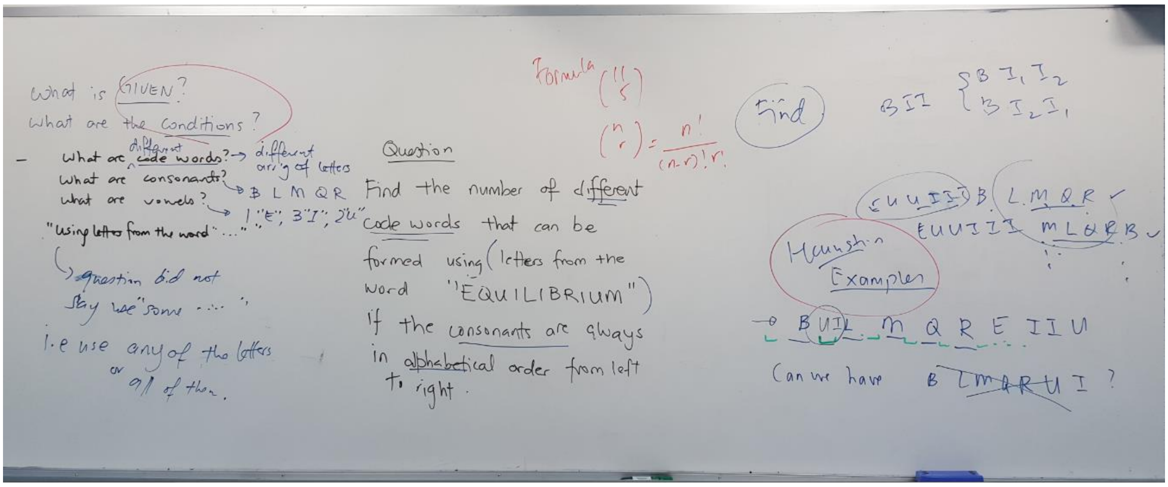


Fig. 7: Critical probing questions arising from the Talk Moves are written on the whiteboard for teachers' emphasis and students' reference

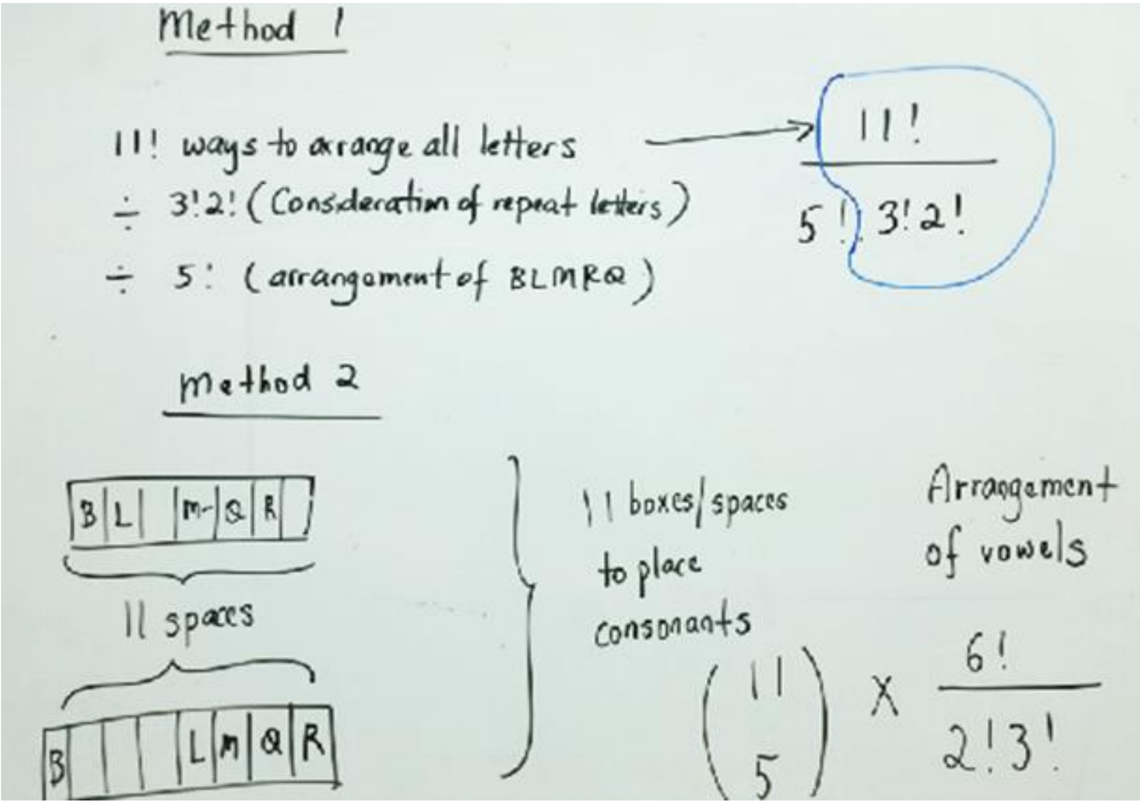


Fig 8: Visual Represenation for the 'Equilibrium' question

Teacher A also demonstrated an approach, with the use of visuals, to help students break down the question into different cases so that they could check that they had examined the problem thoroughly and considered all the possibilities (Fig. 8, Method 2). This was crucial for students themselves to know that they had answered the question completely.

For the remaining problems used in the research study, the student participants were instructed to attempt them individually first before discussing them in their table groups. In tackling the '7-digit number' problem (Fig. 9), Teacher B used a diagram to help students reconcile the given

The digits 1, 2, 3, 8, 9 are used to form a 7-digit number that satisfy one of these guidelines, or both:

(I) The first two digits must be 1 and 2, in some order;

(II) The last two digits must be 8 and 9, in some order.

How many different 7-digit numbers can be formed?

Fig. 9: *The '7-digit number' problem*

information, question requirements and their prior knowledge into categories that were meaningful within recognised problem solving frameworks such as Polya and Conway's four-stage model (Polya & Conway, 2004) and the Paul's Wheel of Reasoning (Paul, 1992) (Fig. 10). This

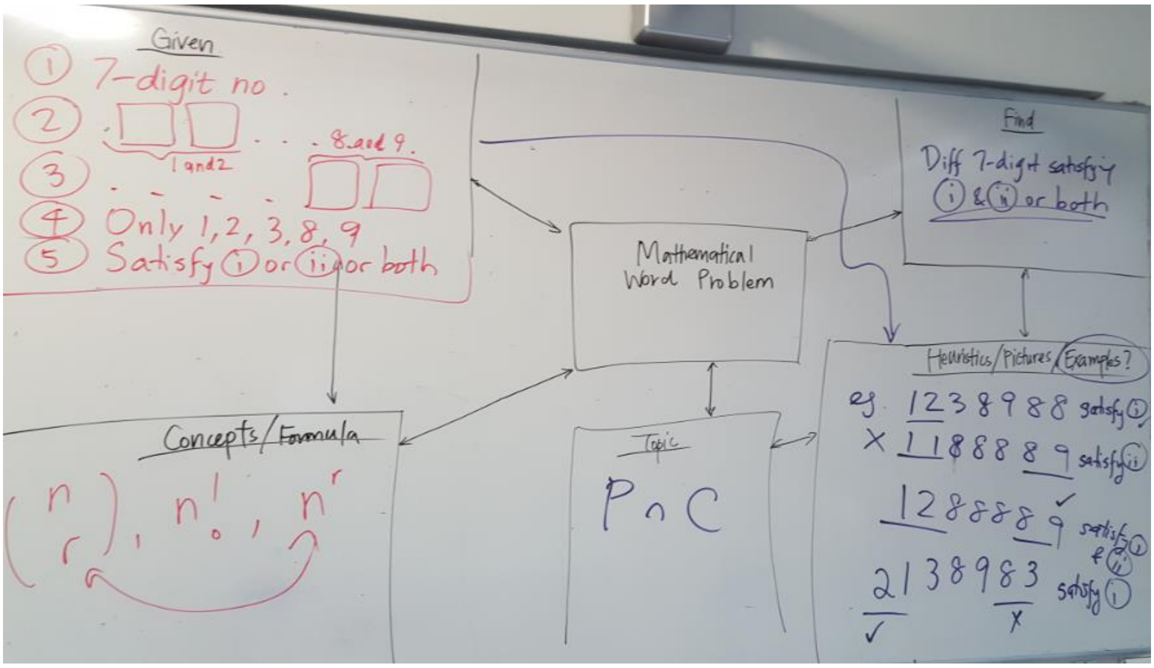


Fig. 10: Diagram with the given problem in categories to facilitate problem solving

strategy was decided upon during the pre-lesson research team's discussion as it could provide greater support in helping the low progress students to make sense of the problem, as better understanding could increase their chances of solving the problem successfully.

Even though most students were initially unfamiliar with the use of a visual diagram to capture useful information and organise their thinking process, they were able to unpack the question correctly through the use of question prompts (Fig. 11).

Speaker	Lesson Transcript	Talk Moves
Teacher B	Now, let's look at this (referring to the diagram), **which part do you think is the easiest to fill up? … What information do you want to write in first?**	**Initiate discussion**
Student	Digits I use to form a 7-digit number.	
Teacher B	Forming a 7-digit number. **Do you think it's here, or there?** Ok, there is 'given' ok. What else? **What other thing is given in the question?**	**Seek Clarification** **Probe for reasoning or evidence**
Student	First two digits must be 1 or 2.	
Teacher B	First two digits must be 1 or 2. Ok? Good. How about you? **What's another piece of information?**	**Probe for reasoning or evidence**
Student	Must be 8 or 9.	
Teacher B	Ok, must be 8 or 9. What else? Is that all the information that we have in the question? **What else do we have? Hmm, there's still something else that's not given, that you've not mentioned.**	**Probe for reasoning or evidence**
Student	The digits, 1-2-3-8-9.	
Teacher B	You can only use the digits, 1-2-3-8-9. Great! **Are there any more things in the question? Anything else that's given?**	**Probe for reasoning or evidence**

Fig. 11: Excerpt showing question prompts that aid in unpacking the problem

There was also a clear emphasis on guiding students to explore the use of problem heuristics and strategies by using analogies to explain main concepts (Fig. 12). As far as possible, Teacher B tried to work on the examples mentioned by students to develop the lesson (Fig. 13). Throughout, we saw a pervasive usage of talk moves by Teacher B to encourage students to participate in classroom talk.

Speaker	Lesson Transcript	Talk Moves
Teacher B	Do you have any examples that you think would satisfy the condition that the first two digits must be 1 and 2 in some order?	
Student	1-1-1-8-8-8-8-9	
Teacher B	The question says, 'It must be 1 and 2, in some order'. So what does it mean by 'must be'? So it's definitive right? So if I say that today I want to watch a movie with Melvin and Ezekiel. **So what do I mean when I say I want to watch a movie with him and him?** So coming back to the question, does 1-1-1-8-8-8-8-9 show 1 and 2 in the first two positions?	**Probe for reasoning or evidence**
Student	No.	

Fig. 12: Excerpt showing the use of analogies

Interestingly, students were either reluctant to employ visuals or were unaware of when to employ these aids strategically. However, they could see how these representations (such as the Box diagram and the Venn diagram) could make a difference to the solution process when these

Speaker	Lesson Transcript	Talk Moves
Teacher B	**So can 1-1-1-8-8-8-8-9 be among the letters you consider in your answer to the question?**	**Seek Clarification**
Student	[inaudible] last two digits.	
Teacher B	ahh…, so even though the first two digits are not 1 and 2, but the last two digits are 8 and 9, in that order. So this does not satisfy condition (i), but it satisfies condition (ii) right? … Ok.. What are some of the other possibilities? Someone from the back said 1-2-8-8-8-8-9. **Why is 1-2-8-8-8-8-9 considered a different case from the first two?**	**Probe for reasoning or evidence**
Student	Because it satisfies both.	
Teacher B	Yes, because it satisfies both. **What about this? 2-1-3-8-9-8-3. Is this ok?** …	**Seek Clarification**

Fig. 13: Excerpt showing the use of examples

strategies were highlighted by Teacher B. The students were able to solve the problem once they were able to make a difference to the solution process when these strategies were highlighted by Teacher B. The students were able to solve the problem once they were able to relate to the given problem in a visual way, without requiring detailed explanations (Fig. 14).

Speaker	Lesson Transcript	Talk Moves
Teacher B	So whenever you want to form a 7-digit number in P and C, **what is the method that most people use?**	**Initiate discussion**
Student	7 factorial…[inaudible]	
Teacher B	[writing on whiteboard] They do this right? What is this? [referring to what was drawn on the whiteboard] What is this thing called?	
Student	Box.	
Teacher B	The box method right? … Now, how are we going to solve the problem? … [Walking over to a student] Let me look at yours. Ok, this guy says 2 factorial times 5 choose 1, times 5 choose 1, times 5 choose 1, times 5 choose 1, times 5 choose 1. **Can you explain how you get it?**	**Seek Clarification**
Student	2 factorial as it can be 1 and 2 or 2 and 1…	

Speaker	Lesson Transcript	Talk Moves
Teacher B	**Now what about the remaining 5 letters?**	**Probe for reasoning or evidence**
Student	From five, choose 1 box, then basically 5 means there are 5 digits, 1-2-3-8-9, and out of that 5, 5 digits choose 1.	
Teacher B	**Ok, what about the others?**	**Probe for reasoning or evidence**
Student	Student: So you bring the same logic over.	

Fig. 14: Excerpt showing how a student solved the problem using the Box method

It is also surprising that students appeared comfortable with the use of symbolic cues when these were applied in the context of a Venn diagram (Figs. 15, 16).

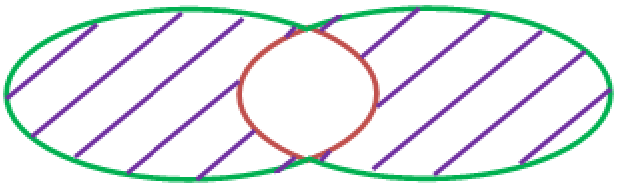


Fig. 15: Symbolic cues for depicting different regions in a Venn diagram

Speaker	Lesson Transcript	Talk Moves
Teacher B	**Can anyone tell me what the diagrammatic representation is, for Case I or Case II or both?** [Venn Diagram] **Which is the area that I want? Is it the crescent moon? Or two crescents? Or is it the peanut? Or is it the almond?** Anyone? [Walking over to a group] Which one do you think it is?	**Initiate discussion** **Seek Clarification**
Student	Peanut.	
Teacher B	The peanut, correct. So this is what we want to find. So how do we find this set? 1 + 2 or minus? Ok, correct. I'm not going to work out the answer for you, but you can see very clearly this gives you the answer.	

Fig. 16: Excerpt showing the use of symbolic cues during discussion on Venn Diagram

Hence, this suggests that teacher facilitation and the use of talk moves are essential in creating a crucial scaffold through visual and symbolic representations for these strategies to be harnessed effectively in mathematical problem solving.

Finally Teacher C brought the student participants through the 'Mathematics' Problem (see Fig. 17).

A student wants to form a 5-letter code word using letters from the word MATHEMATICS.

Find the number of different 5-letter code words such that

(i) all letters are distinct letters,
(ii) exactly three letters are vowels.

Fig. 17: The 'Mathematics' problem

In part (ii) of the problem, the main challenge lies in the construction of the case definition (the criteria that need to be fulfilled to be considered a case) that is elegant and sound. What students also find challenging is finding a self-directed way to check whether their cases are exhaustive or not. Also of importance is how the different cases should be mutually exclusive to each other. To facilitate the thinking process, Teacher C used a systematic method of 'listing' and 'talking aloud' with the students and complemented these with visuals (see Fig. 18) accompanying relevant topical terminologies (including VR – Vowels Repeat, CD – Consonants Distinct).

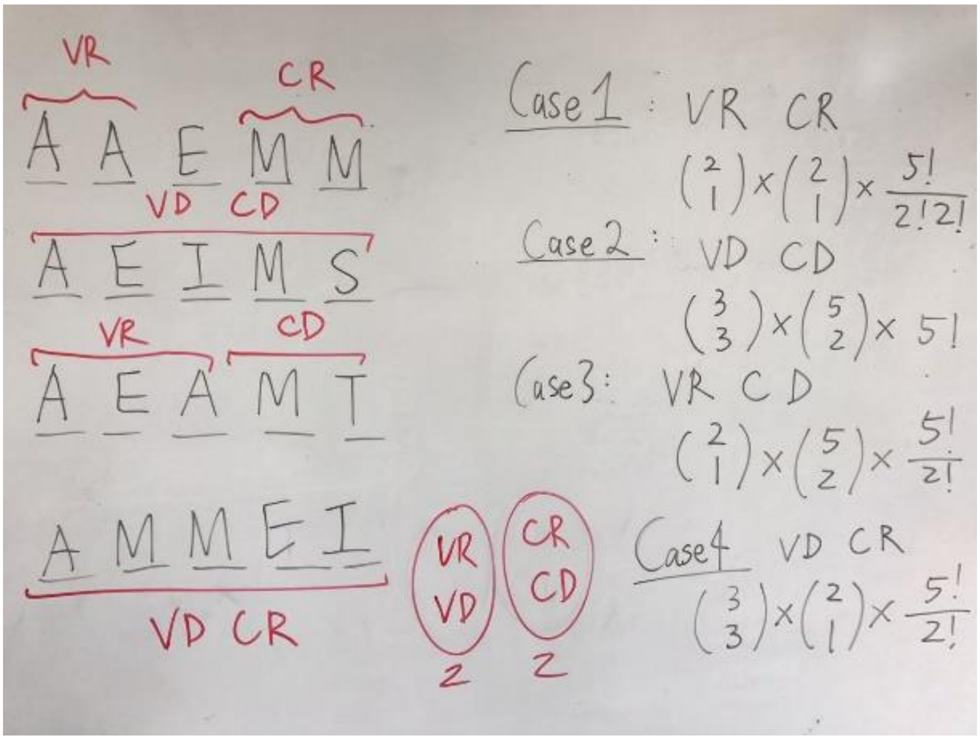


Fig. 18: Diagram that shows how visuals were used to make thinking more explicit in problem solving

The use of talk moves could be seen from the transcript, in the way Teacher C led students into coming up with examples for each of the cases (see Fig. 19).

Speaker	Lesson Transcript	Talk Moves
Teacher C	Now, all your queries are from part 2, so let's deal with it now. Part 2 says that the 3 letters must be vowels, so I must choose another 2 consonants. … **Just freely give some examples for 5-letter code words. Give me examples of code words with 3 vowels and 2 consonants.**	**Initiate Discussion**
Student	A-A-E-M-M	
Teacher C	**Good, next one, Yu Lin, anymore?**	**Probe for reasoning or evidence**
Yu Lin	A-E-I-M-S	
Teacher C	**Good.**	**Probe for reasoning or evidence**
Student	A-E-A-M-T	
Teacher C	Mmm, yes fair enough. **Cindy?**	**Probe for reasoning or evidence**
Cindy	A-E-I-H-C	
Teacher C	Ok can, more or less we have some examples, so let's think of ways to approach this question. The examples listed are actually a wide range. Anything that we noticed in this example? These are the vowels, repeated, this has a vowel repeat, while here there is a consonant repeat. (Mentioned the remaining cases.) So listing examples doesn't only help us to get a sense of how we approach the questions, it also helps to consolidate all the spaces and how to group them. The cases should probably be from some of your examples. **Melvin, would you like to try?**	**Probe for reasoning or evidence**
Melvin	Vowels repeat.	
Teacher C	The vowels repeat, which means the only way to get a vowel repeat is to get A-A. **Let's have the second case. Wan Chin.**	**Probe for reasoning or evidence**
Wan Chin	Vowel distinct, consonant distinct.	
Teacher C	Ok, how many ways? Vowels must be distinct, vowels distinct means…	

Fig. 19: Excerpt showing the use of talk moves to list examples in order to identify the various cases involved

After the lesson, a teachers' focus group discussion was conducted to discuss the students' learning process and surface students' learning gaps.

Results

Data Analysis

How do teacher talk moves support students' learning of Mathematics?

A total of three teachers were involved in the Phase 1 implementation. Of the three, only Teacher B conducted the research lesson in the subsequent Phase 2 stage. The students who participated in the Phase 1 and 2 studies were low ability students who had not fared well in the most recent examinations. Using an interpretative qualitative approach, the lesson transcripts were analysed to ascertain the types of talk moves that were employed and the frequency with which each type of talk move was used during the lessons. Fig. 20 shows the tabulation of talk moves used by each of the teachers, A, B and C, while Fig. 21 shows the tabulation of talk moves used in each phase.

Specific Talk Moves	**Teacher A**		**Teacher B**		**Teacher C**		**Total**	
	Freq	**%**	**Freq**	**%**	**Freq**	**%**	**Freq**	**%**
Seek Clarification	12	30.8	18	22.0	2	22.2	32	24.6
Revoice for Verification	2	5.1	7	8.5	0	0	9	6.9
Ask student to restate another student's contribution	0	0	0	0	0	0	0	0
Probe for reasoning or evidence	19	48.7	30	36.6	2	22.2	51	39.2
Challenge students' statement or assumption	4	10.3	10	12.2	1	11.1	15	11.6
Elicit students' views on other students' ideas	2	5.1	17	20.7	3	33.3	22	16.9
Guide students to build on other students' contribution	0	0	0	0	1	11.1	1	0.8
Get students to summarise /consolidate	0	0	0	0	0	0	0	0
Total	39	100	82	100	9	100	130	100

Fig 20: Tabulation of the specific talk moves applied by each teacher

As shown by the analyses of individual teachers in Fig. 20, *Probe for reasoning or evidence* was used most frequently by Teachers A and B. This particular move, however, did not register the highest utility by Teacher C, probably because there was less need to delve for more evidence of reasoning given the earlier efforts of Teachers A and B and the fact that the students were able to provide sensible and reasonable responses during previous segments of the lessons.

Seeking Clarification remains one of the most commonly used talk moves. All three teachers used this talk move in comparable amounts. This may point to the correlation between the usage of this talk move with the ability profile of the participants and the individual instructional style of the teachers.

It is also interesting to note that both Teachers B and C allocated a fair proportion of question prompts to the talk moves *Eliciting students' views on other students' ideas* and *Guiding students to build on other students' contribution*. While this preference could be attributed to individual teaching styles, this move can also be interpreted as the teachers' attempt to move away from direct instruction and encourage the participants to listen and respond to their peers, or as a means to engage the students more actively as the lesson progressed.

Specific Talk Moves	Phase 1 (Teachers Involved: A, B and C)		Phase 2 (Teachers Involved: B)		Total	
	Freq	%	Freq	%	Freq	%
Seek Clarification	18	26.1	14	23.0	32	24.6
Revoice for Verification	5	7.2	4	6.6	9	6.9
Ask student to restate another student's contribution	0	0	0	0	0	0
Probe for reasoning or evidence	32	46.4	19	31.1	51	39.2
Challenge students' statement or assumption	7	10.1	8	13.1	15	11.5
Elicit students' views on other students' ideas	6	8.7	16	26.2	22	16.9
Guide students to build on other students' contribution	1	1.4	0	0	1	0.8
Get students to summarise /consolidate	0	0	0	0	0	0
Total	69	100	61	100	130	100

Fig. 21: Tabulation of the specific talk moves applied during each Phase

From Fig. 21, *Seek Clarification* and *Probe for reasoning or evidence* were undoubtedly the most frequently used talk moves in both Phases 1 and 2. A possible reason for this was how these two moves were able to target the specific misconceptions or learning required by the students for this topic.

Excerpt 1		Talk Moves
Student	I thought that there were 11 letters, so 11! ways to arrange all the letters, divide by 3!2! because of U and I, then divide by 5! **because we need to arrange the 5 consonants.**	**Seek Clarification**
Teacher	But how many ways are there to arrange the consonants?	
Student	Only 1 way because need to be in alphabetical order?	
Teacher	So do we still need to divide by 5! **Why do we need to divide by 5!?**	**Probe for reasoning or evidence**
Student	So that each arrangement of the consonants will not be counted separately?	
Teacher	**Can another student elaborate this further?**	**Ask student to restate another students' contribution**

Excerpt 2		Talk Moves
Teacher	**Can you explain your answer?**	**Revoice for verification**
Student	Rearranging of the 6 vowels, then divide by 2!3! because of the repeating vowels.	

Excerpt 2		Talk Moves
Teacher	**Is 6! ways of arranging the vowels the same as number of ways to separate the vowels? How many ways are there to separate the vowels?** Discuss!	**Challenge students' statement or assumption** **Elicit students' views on other students' ideas**

Excerpt 3		Talk Moves
Teacher	**Does your example 1188889 show (the numbers) 1 and 2 in the first two positions?**	**Seek clarification**
Student	No.	
Teacher	**So can this be among the numbers that you want? 1 and 2 in some order?**	**Probe for reasoning or evidence**
Student	Yes because the last two digits are 8 and 9.	

Fig. 22: Excerpts of classroom transcript during group discussion, demonstrating different specific talk moves

The excerpts in Fig. 22 further illustrated how the talk moves *Seeking Clarification* and *Probing for reasoning or evidence* were used during lesson to help students verbalize explicitly what they were thinking in order to increase their accuracy and precision in reasoning.

As seen from the excerpts, the initial responses from the students were usually brief or not specific. A plausible explanation for this is that students only possessed an intuitive idea of the solution approach, but did not have the correct mathematical language and sufficient conceptual understanding to support their ideas thoroughly. Through further probing, students were guided to give a more detailed elaboration.

Besides the conducting teachers, it was later found, during the post-lesson discussions, that the talk moves, *Revoice for verification*, *Challenge students' statement or assumption*, and *Elicit students' views on other students' ideas,* were also used frequently by the teachers facilitating the table group discussions. These talk moves were useful in addressing certain uncommon misconceptions brought up by some participants in the groups, and since the general profile of the lesson participants was weak, these concerns could not be discussed with the rest of the class as this might have created grounds for further confusion. Unfortunately, no written or recorded evidence documenting the usage of these talk moves during the table group interventions were captured for a more detailed analysis. Nevertheless, it is clear that the usage of the three talk moves *Revoice for verification*, *Challenge students' statement or assumption*, and *Elicit students' views on other students' ideas* were not sufficiently captured by the transcripts as these were also applied by the teacher facilitators when they handled the discussions within the respective table groups.

How does use of visual representation support students' learning of Mathematics?

Due to the absence of students for one or other of the pre- or post-tests, only 20 out of the 30 students test results were analysed. One mark was assigned each time a student used any visual forms or representations to answer a pre- or post-test item. The scoring for the students' use of visuals was added up as shown in the table below. The test score (based on the students' mathematical computations) of a student was also computed by adding up his/her scores for all the questions in the test. Fig. 23 reports the summary statistics of the pre- and post-test scores for all 20 students participants who attended the research lesson in Phase 1.

	Scoring for students' use of visuals		Score on the test	
	Pre	Post	Pre	Post
Mean	1.5	2.5	1.1	5.7
Standard Deviation	1.7	2.9	1.1	3.6
Range	7	11	3	10
Minimum	0	0	0	0
Maximum	7	11	3	10

Fig. 23: Analyses of pre and post test

Due to the small sample size, a Hedges' *g* statistic (Hedges, 1981) was computed to find out the effect sizes for the scoring of the use of visuals and the test. However, the computed result showed little effect based on the test scores (−1.69 for the scores on the test and −0.412 for the scores for the use of visuals). This was due to the great disparity in terms of students' improvement in test scores after the research lessons. While some students made tremendous improvement after the intervention, as shown by their test scores and the quality of their work, which showed clearly presented solutions complete with diagrams and heuristics, there were also some students who did not make any improvement at all. This could be due to the presence of other factors which could have possibly hindered their understanding of this topic, and these issues were not surfaced during the lesson.

Although there is little indication of an effect through quantitative data analysis, a closer look at some of the students' work in the pre- and post-tests (written notes, annotations, and other workings) provided some qualitative evidence of a positive change in some students' approach to solving the problems. Some students made an effort to include more visuals to explain their thinking processes, heuristics usage, and literacy skills.

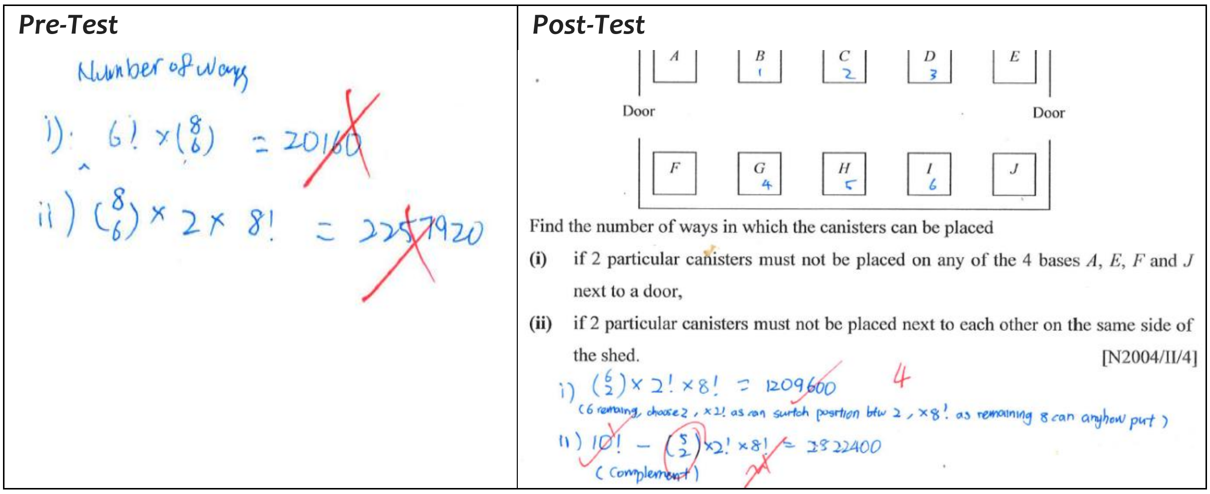


Fig. 24: Improvement in student's Mathematical Literacy in articulating ideas logically in written form

Fig. 24 shows the pre- and post-test solution of one participant of the Phase 1 research lesson. After the intervention lesson, this student clearly showed greater awareness of the P&C methods such as the use of the complement method (see solution for (ii)). This student also succeeded in articulating his reasoning clearly using words in brackets (see solution for (i)) to justify his solutions.

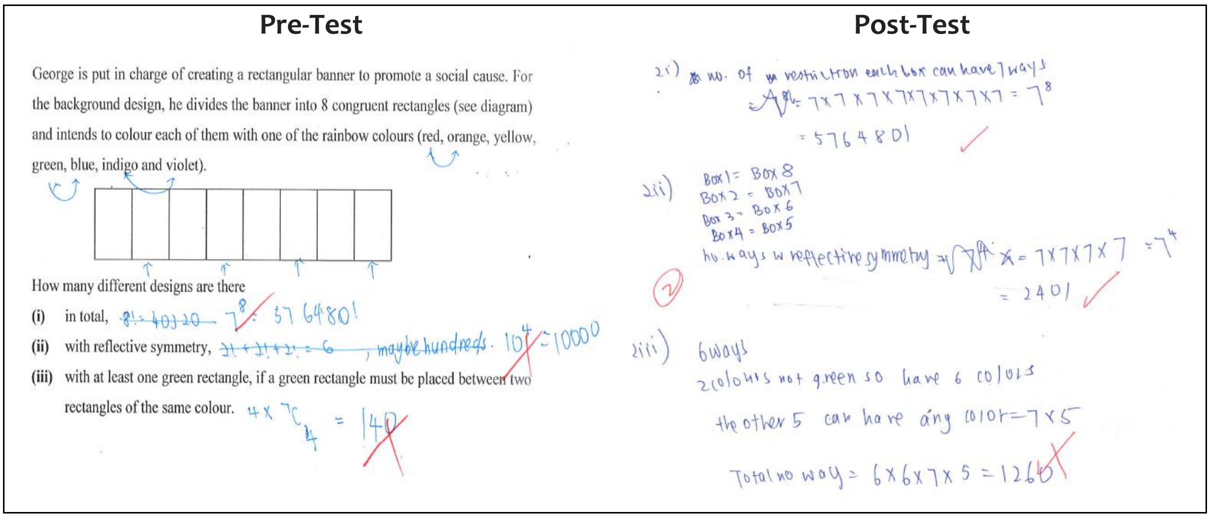


Fig. 25: Increased clarity in the thinking process, evident through the planning and articulation of ideas

The student's work featured in Fig. 25 showed an earnest effort to convey thinking more clearly and precisely as the intermediate steps showed clear connections with key P&C concepts such as the Multiplication Rule. The student also tried, in the solution to (ii), to communicate his thinking on how reflective symmetry would affect the colour choices in each of the boxes, by drawing links between the colour choices in the first and eighth box, between the second and the seventh box, and so on. Even though he did not get (iii) correct, he was aware that this was a complicated problem that required him to analyse the situation in parts, as suggested in his explanations.

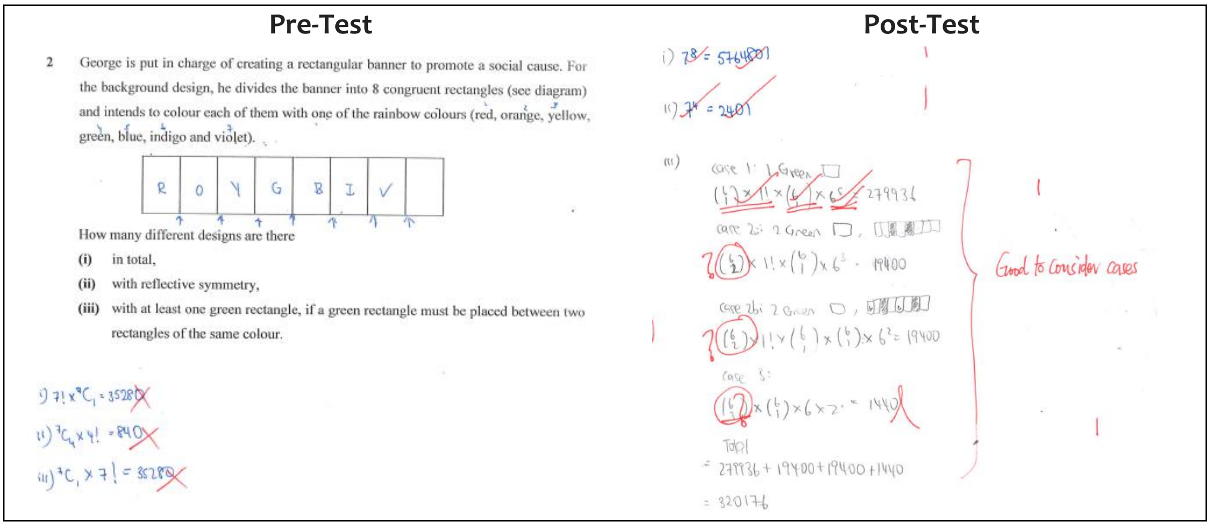


Fig. 26: Improvements in expressing ideas concisely by using visual and classification by cases

From the pre- and post-test script provided in Fig. 26, the student showed his preference for succinctness throughout. Even in (iii) where he approached the question using a combination of visual representations and the method of classification by cases, he was able to complement his working for each of the cases with visuals, which helped him to bring across his arguments with brevity and clarity.

Pre-Test	Post-Test

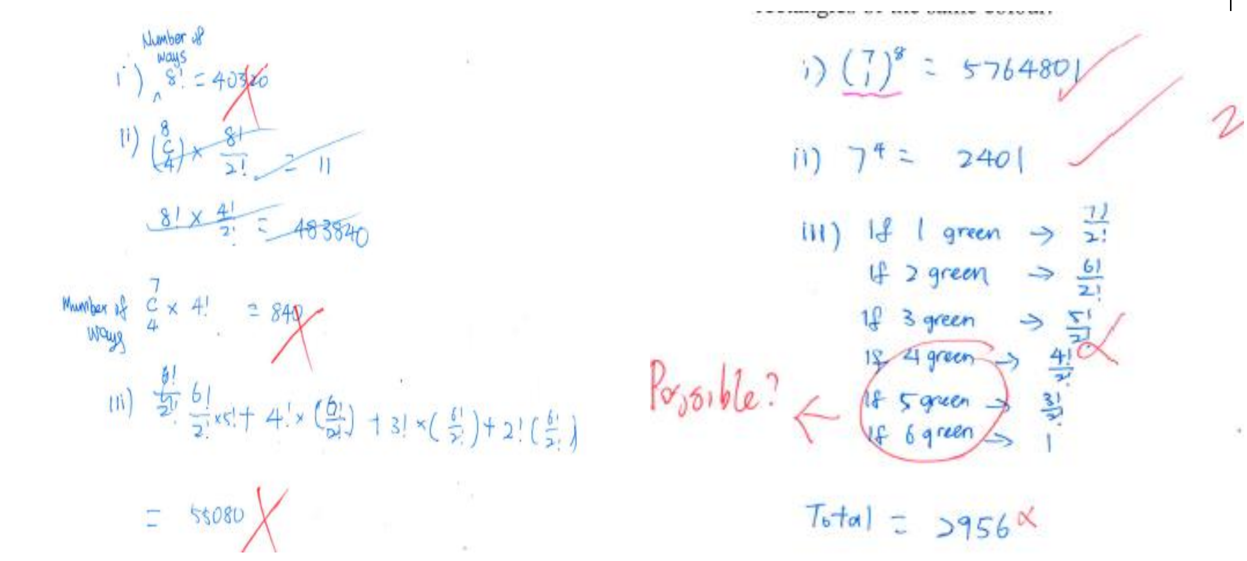


Fig. 27: Use of classification as a concrete heuristic to analyse the problem

The pre- and post-test solutions in Fig. 27 showed a marked difference in the presentation of solutions which involved the method of classification by cases (see (iii)). Unlike in his earlier response, where there were no details given with regard to how the classification by cases came about, the student was now able to articulate, after the intervention lesson, how he divided the problem into cases, i.e. by considering the number of green coloured rectangles.

Students' Perspectives

After the lesson, the students (20 from Phase 1 and 25 from Phase 2) gave feedback on how they perceived the lessons to have impacted their learning of P&C, and the extent to which the use of visual representations mediated by talk moves had enhanced their learning of the topic and developed their mathematical literacy skills. The findings summarised in Fig. 28 revealed that the interventions resulted in an improvement in the participants' ability to understand the content, as the opportunities to 'think aloud' due to the use of talk moves, helped them see the necessary links between and within mathematical ideas. The lessons also managed to heighten the participants' awareness of the different approaches to analysing problems.

Focus Question	**Observations**	**Students' Feedback – sample responses**
How does use of visual representation mediated by talk support students' literacy skills?	Raises students' awareness of • seeing and making linkages among mathematical ideas • exploring different approaches to analysing problem	'Can see the **different ways to approach** questions and how others came to the final solution' 'Very engaging, step by step analysis, the solving problem steps (**identifying the given and what to find**)'

Focus Question	Observations	Students' Feedback – sample responses
		'Lesson more interesting, makes me think more about **various ways to solve** problems' 'I like discussing the tough parts together and raising **different interpretations** of the questions'
How does use of visual representation mediated by talk support students' learning?	Motivates students to think	'Interactive, enables me **to think better**' 'Question were gone through in super incredible detail and time was allocated for group discussion which **forced us to think**' 'Engaging, keeps us **actively thinking** about how to solve the questions' '**Forced us to really listen** and absorb at one shot → helps **keep the thinking connected** (especially since students don't usually revise immediately after learning)'
	Enhance students' perception of their conceptual understanding	'The lesson was very interesting and I revised for P&C and managed to strengthen my concepts' 'Understand P&C better through the enrichment'

Fig. 28: Summary of students' feedback

Discussion

Pedagogical Implications

Effectiveness of Intervention Sessions

It is safe to say that the effectiveness of the intervention sessions depended a great deal on the teachers' comfort level and proficiency in the use of talk moves in the classroom to elicit responses that revealed students' understanding of the topic. Even though all three teachers had different teaching styles, they were able to leverage their knowledge of the question frames to direct discussions to help students develop a better understanding of the concepts, requirements of the questions and the solutions. Besides clarifying understanding, addressing various misconceptions and prompting deeper thinking, the teachers succeeded in demonstrating and guiding students along to unpack the questions and think systematically. Through the teachers' effective use of visuals and board work, students were able to see how the heuristics (such as creating an exhaustive list of cases, drawing a picture, listing examples) could be applied to good effect in problem solving. In particular, the students, when working on the 'EQUILIBRIUM' problem, were able to come up with a more straightforward method that also helped them to make sense of the concept of the grouping of repeat cases. Furthermore, the research lessons also revealed that open and generic questions posed by the teachers were very effective in getting students to articulate their thought processes, as evidenced by how these questions were able to trigger the students to ask more open questions. By getting students to challenge questions and answers based on what was raised and discussed, students also had an alternative way to clarify their thinking.

Professional Development for Teachers

This study provided valuable opportunities for the teacher instructors and observers to learn from one another with regard to board work and questioning techniques. By observing how each teacher built upon common questions to probe and delve deeper into the reasoning and thinking behind some of the students' responses, it became clear how the extent of teacher preparation, the familiarity with talk moves and the pedagogical subject content knowledge of the teacher had played a crucial part in driving productive classroom discussions. The focus group discussions also provided a meaningful platform for the teachers involved to share their individual perspectives of teaching, and their beliefs and perspectives on how interaction in the classroom should be managed in order for learning to take place.

The research study was also informative for the teacher instructors because of the presence of observers, who could offer feedback which the teacher instructors might not even have been aware of. More importantly, the observers could provide valuable insights into students' thinking during explanations and discussions – which could present, in themselves, another valuable source of data which could help uncover underlying learning mechanisms in the classroom. The lessons also made the teacher instructors more aware of their use of language, as there was a greater attempt in bringing up more technical terms in the discussions so that students could be primed in mathematical thinking and syntax – so that they would grow to be comfortable using such terms confidently during discussions.

The session also revealed areas for improvement, such as the need for teachers to avoid asking too many questions at once and allowing for more wait-time after posing questions. More importantly, it is imperative that teachers continue to role model proper usage of mathematical language during instruction so as to promote and encourage the use of precise and accurate academic language in the classrooms.

Moving forward, the findings from our study can be used to benefit other colleagues in the Mathematics Department in their teaching of P&C through department sharings. The various concrete research artifacts such as videos, transcripts, students' reflections and teachers' feedback derived from our study will be compiled into a reference resource for department teachers. The teachers will have access to the videos and transcripts of recorded lessons, which can provide deeper insight into the interplay between the different types of talk moves and their frequency and how these can be artfully modified to suit different teaching styles without compromising the productiveness of teacher-student discussions. Another useful resource that will be made available is the collation of students' questions and errors surfaced from the research lessons. This information will be helpful to other teachers in the department, should they be interested in exploring the use of talk moves and visual representations, as they can use the collation as a basis to uncover students' thoughts so that thinking can be made more purposeful and misconceptions can be more easily identified and rectified. In addition, the collation can also serve as a reference for colleagues to improve the quality of their feedback when responding to students' queries during classroom discussions. Furthermore, teachers in the research team can also invite their colleagues to visit their classes to observe how classroom discussions are facilitated using strategies identified from this study. Such measures can help to promote more interactive and engaging teaching while scaling up the professional development of colleagues so that students beyond the scope of our study can also benefit.

Indeed, there is always something worthwhile learning when teachers collaborate and work with educational professionals. Through the consultation sessions with the ELIS consultants, the research team gained a deeper awareness of the various details required at each stage of the research process, and learnt how to plan and implement the research to take into account practical considerations, research and educational outcomes. The team was also inspired to think more deeply into the focus, planning and implementation of the lessons, following sharings by the ELIS consultants on their past research mentoring experiences. Beyond the department, the research team has presented their findings both internally (such as on the 2016 College's Professional Sharing Day) and externally (such as on the 2016 *East Zone COE T&L Professional Sharing Day*). Indeed, the team has grown in terms of their capacity to handle research and their ability to employ talk moves to benefit actual everyday classroom teaching and instruction.

Conclusion

Our study has shown that a teacher's choice of talk moves in teaching may vary depending on the learning profile and attitudes of the students, as well as the purposes for which the talk moves are intended. In handling the low ability students, our research has indicated that the most frequently

Focus Area		Talk Moves	Frames for Responding
1	Voicing and clarifying students' ideas	Seek Clarification	• What I mean is …
		Revoice for verification	• Yes, that's right.
2	Listening closely to other students	Ask student to restate another student's contribution	• I think what X was saying is …
3	Deepening individual students' reasoning	Probe for reasoning or evidence	• The way I could tell was because …
		Challenge students' statement or assumption	• I guess another way to look at/explain it is …
4	Engaging with each others' reasoning	Elicit students' views on other students' ideas	• I agree with X because …
		Guide students to build on other students' contribution	• I would add that …
5	Consolidating discussion points	Get students to summarise/ consolidate	• I think the main point(s) about X is/are …

Fig. 29: Teacher Talk Moves and Example Frames for Responding from ELIS's *Opening Up Talk for Learning in Subject Classrooms* course

used talk moves by all three teachers are *Probing for reasoning* and *Challenging students' statement or assumption*, which fall within the focus area of *Deepening students' reasoning*. Another talk move that is also used quite regularly when teaching weaker students is *Seeking clarification*, which comes under the focus area of *Voicing and clarifying students' ideas* (Fig. 29). However, we are unable to ascertain if the choice of talk moves is dependent on the nature of the subject, even though the extensive use of particular types of talk moves over other types in our study seems to suggest this. Perhaps this is an area worth exploring in future studies. Besides this, interested educational researchers may also wish to look into ways to help teachers expand their repertoire of questioning strategies using talk moves so as to better support the use of visual representations that can aid students' problem solving in Mathematics.

The core of Mathematics learning depends on students' problem solving skills. Our study looked into how teachers' role modelling using visuals can support students' understanding of a problem, which in turn can lead onto solving the problem. Further research will be useful to find out how improved students' mathematical literacy skills can have a direct impact on their problem solving skills.

The results from this study also revealed the need to create a culture where students can be more responsive. One way to promote students' participation in classroom talk is to teach them how to frame their responses by using the ELIS framework (Fig. 29). By building up their capacity to handle teacher's questioning prompts, they can benefit more through teacher-student interactions.

From our study, it was found that teacher responses to students' contributions were the most crucial aspect leading to productive classroom discussions. This suggests that the overall quality of classroom talk will also depend on the ability of the teacher to build on the points and concerns raised by students in a meaningful way. Hence further study will be needed to find out the effect of the teachers' subject pedagogical content knowledge on the quality of teacher feedback and how all these can have a bearing on how productive teacher-student classroom talk can be. Also of interest is how distinct styles of teaching might impact student's literacy skills and problem solving processes. In particular, it might be insightful to observe the differences in instruction and the use of talk moves between a more 'structured' teacher, i.e. one who is more consistent and methodological in implementing instructions, and another who prefers a 'free-style' approach and is less inhibited by the need to follow lesson plans strictly. Such findings will have important implications for future research studies on classroom talk.

Acknowledgments

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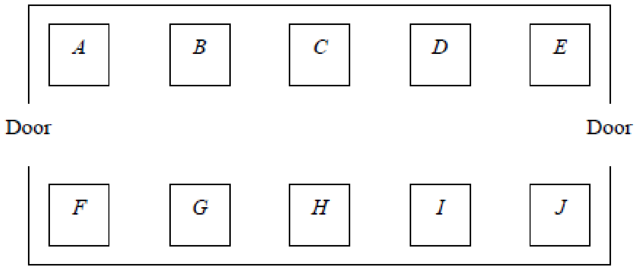
Appendix 1: Pre- and Post-Test Questions

PERMUTATIONS AND COMBINATIONS Pre and POST-TEST

1 A rectangular shed, with a door at each end, contains ten fixed concrete bases marked $A, B, C, \ldots, J$, five on each side (see diagram). Ten canisters, each containing a different chemical, are placed with one canister on each base. In how many different ways can the canisters be placed on the bases?

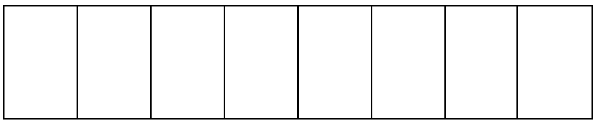


Find the number of ways in which the canisters can be placed

(i) if 2 particular canisters must not be placed on any of the 4 bases A, E, F and J next to a door,

(ii) if 2 particular canisters must not be placed next to each other on the same side of the shed. [N2004/II/4]

2 George is put in charge of creating a rectangular banner to promote a social cause. For the background design, he divides the banner into 8 congruent rectangles (see diagram) and intends to colour each of them with one of the rainbow colours (red, orange, yellow, green, blue, indigo and violet).



How many different designs are there

(i) in total,

(ii) with reflective symmetry,

(iii) with at least one green rectangle, if a green rectangle must be placed between two rectangles of the same colour.

Appendix 2: Students' Feedback

One Student Comment: Expand this to the whole cohort. Confusion between n!, n^r and $\binom{n}{r}$ sorted out very clearly.

What I like most about the lesson?

- Interactive, enables me to think better. Understand P&C better through the enrichment.
- Lesson more interesting, makes me think more about various ways to solve problems.
- I like today's lesson because it is so slow paced that I can catch the content of the lesson, and how insignificant examples can actually solve the question.
- Very engaging, step by step analysis, the solving problem steps (identifying the given and what to find etc.)
- Engaging, keeps us actively thinking about how to solve the questions.
- The lesson was very interesting and I revised for P&C and managed to strengthen my concepts.
- It did clarify some questions that I had about PnC previously since the questions were done at a moderate pace with good explanations.
- Can see the different ways to approach questions and how others came to the final solution.
- I like discussing the tough parts together and raising different interpretations of the questions.

- Question were gone through in super incredible detail and time was allocated for group discussion which forced us to think.
- Forced us to really listen and absorb at one shot => helps keep the thinking connected (especially since students don't usually revise immediately after learning)
- I think the different teachers teaching was good! When you don't understand what one teacher is teaching, there's other teachers to ask/explain it to you!
- I felt like I learnt a lot more in 2 hours rather than over a few disconnected tutorials?;)

What I do not like about the lesson?

- A bit too rushed.
- Long explanations (especially first question) - too long and pace of lesson was very slow, I tend to zone out.
- I don't like today's lesson as I understood the questions, but still feel not confident about tackling the question (guess I need more practice)
- Question 1 was covered too quickly and a bit hard to understand.
- Lesson was too long, maybe can have a break in between :) People started staring into space and it was hard to focus coz too tired. Give 5 min breaks? As long as the student can get some mind rest/gather their thoughts before proceeding to follow with the lesson! :)